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Shifted Modified Bernstein pseudo –spectral collection method for solving differential equations

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Abstract:

In this article, a generalized shifted Bernstein pseudo –spectral collection is developed for solution of the linear equations of fractional order with variable coefficients, under the mixed conditions. These equations are expressed as linear equation by using generalized shifted Bernstein collection method, solution of these equation are approximated, and the approximation solution is obtained. Moreover, some numerical solutions are given to illustrate the accuracy and implementation of the method.

Keyword:

Bernstein polynomial approximation, linear differential equation, collocation method.

Introduction:

Differential equations, which describe how quantities change across or space arise naturally in science, engineering and in almost every field of study where measurements can be taken. Most realistic mathematical models cannot be solved through the traditional pencil and paper techniques providing an excellent means to put across the underlying theory; instead, they must be dealt with the computational methods that deliver approximate solution[1],[2].

In this paper, we consider the numerical solution of linear fractional differential equation of the form

$$\frac{d^m u}{dt^m} - a\frac{d^{\propto} u}{dt^{\propto}} - bu = f(t) \qquad (1-1)$$

Where $t > 0, m - 1 < \propto \leq m$

Subject to the initial conditions

$$u^{(j)}(0) = Cj, j = 1, 0, \dots, m-1 \qquad (1-2)$$

Where $Cj, j = 1, 0, \dots, m-1$, are arbitrary constants and u(t) is assumed to be a causal function of time, i.e., vanishing for t < 0.

The fractional derivatives is considered in the caputo sense, the general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various Reponses [5],[7].

polynomial have played a center role in approximation theory and numerical analysis for many years.

Bernstein polynomials have many useful properties such as the positive, the continuity, recursion's relation, symmetry and unity partition of the basis set over interval [0, 1], these polynomials have been utilized for solving several equation by using various numerical method [3],[8],[9].

$\underline{\boldsymbol{Definition\ (1-1)}}$

A real function f(x), x > 0, is said to be in the space $C_{\mathcal{M}}, \mathcal{M} \epsilon R$ if there exists a real number (P > $\mathcal{M}$), such that $f(x) = x^p f_1(x)$, where $f_1(x) \epsilon C\ [0, 0]$ and it is said to be in the space

$C_{\mathcal{M}}^{m}\ iff\ f^{(m)}\ \epsilon\ C_{\mathcal{M},}\ m \epsilon N$,[4].

$\underline{\boldsymbol{Definition\ (1-2)}}$

The Riemann –liouville fractional integral operator of order $\propto \geq 0$ of a function $f\ \epsilon\ C_{\mathcal{M}}, \mathcal{M}\ \geq\ -1$ is defined by:

$$J^{\propto} f(x) = \frac{1}{\sqrt{(\propto)}} \int_0^x (x-t)^{\propto -1}\ f(t)\, dt \qquad (1-3)$$

$$\forall \propto > 0, x > 0$$

$$J^0 f(x) = f(x) \qquad (1-4)$$

Properties of operator $J^{\propto}$ can be found [25], we mention only the following for $f\ \epsilon\ C_{\mathcal{M}}, \mathcal{M}\ \geq\ -1\ \ \propto, \beta\ \geq 0\ and\ \boldsymbol{\lambda} >\ -1$:

1- $J^{\propto} J^{\beta}\ f(x) = J^{\propto + \beta} f(x)$

2- $J^{\propto} J^{\beta}\ f(x) = J^{\beta} J^{\propto}\ f(x)$

3- $J^{\propto} J^{\lambda} = \frac{\sqrt{(\lambda+1)}}{\sqrt{(\propto+\lambda+1)}}\ x^{\propto+\lambda}$

The Riemann –liouville derivative has certain disadvantages when trying to model real –world phenomena with fractional differential equations [6].

Definition (1 − 3)

The fractional derivative of f(x) in the caputo sense is defined by:

$$D^{\propto}f(x) = J^{m-\propto}D^{m}f(x) = \frac{1}{\sqrt{(m-\propto)}}\int_0^x (x-t)^{m-\propto-1}f^{m}\,(t)dt \qquad (1-5)$$

For $m - 1 < \propto \leq m, m \,\epsilon\, N, x > 0, f \,\epsilon\, C_{-1}^{m}$

Also we need here two of its basis properties [6].

Lemma (1 − 1)

If $m - 1 < \propto \leq m, m \,\epsilon\, N \; and \; f \,\epsilon\, C_{\mathcal{M}}^{m}, \mathcal{M} \geq -1 \; then$

$$D_*^{\propto}J^{\propto}\, f(x) = f(x) \; and$$

$$J^{\propto}D_*^{\propto}f(x) = f(x) - \sum_{K=0}^{n-1} f^{(k)}(O^+)\,\frac{x^k}{k!}, x > 0$$

Definition (1 − 4)

Generalized Bernstein basis polynomials can be defined on the interval [0, L] by

$$P_{i,n}(x) = \frac{1}{(L)^n}\binom{n}{i}(x)^i(L-x)^{n-1};$$

For convenience, we set $P_{i,n}\,(x) = 0 \; if \; i < 0 \; or \; i > n$

We give the properties of the generalized Bernstein basis polynomials in the following list:

(a) Positivity property
$P_{i,n}\,(x) > 0$ is hold for all $i = 0, 1, \dots, n$ and all $x \,\epsilon\, [0, L]$

(b) Unity partition property

$$\sum_{i=0}^{n} P_{i,n}(x) = \sum_{i=0}^{n-1} P_{i,n-1}(x) = \cdots = \sum_{i=0}^{1} P_{i,1}(x) = 1$$

(c) Recursion's relation property
$P_{i,n}(x) = \frac{1}{L}\left[(L-x)P_{i,n-1}(x) + xP_{i-1,n-1}(x)\right]$

$\underline{Definition\ (1-5)}$

Let y: [0,L] →c be continuous function on the interval [0, L]. generalized Bernstein polynomials of degree n are defined by

$$B_n(y;x) = \sum_{i=0}^{n} y\left(\frac{L}{n} i\right) P_{i,n}(x)$$

2 − Fundamental Relations

$\underline{Theorem\ (2-1)}$

On the interval [0,L], any generalized Bernstein basis polynomials of degree (n) can be written as a linear combination of the generalized Bernstein basis polynomials of degree n+1

$$P_{i,n}(x) = \frac{n-i+1}{n+1} P_{i,n+1}(x) + \frac{i+1}{n+1} P_{i+1,n+1}(x)$$

$\underline{Proof}$

By using definition (1 − 4), we have

$$\frac{x}{b} P_{i,n}(x) = \frac{x}{(L)^{n+1}} \binom{n}{i} (x)^i (L-x)^{n-i}$$

$$= \frac{\binom{n}{i} / \binom{n+1}{i+1}}{(L)^{n+1}} \binom{n+1}{i+1} (x)^{i+1} (L-x)^{n+1-(i+1)}$$

$$= \frac{\binom{n}{i}}{\binom{n+1}{i+1}} P_{i+1,n+1}(x) = \frac{i+1}{n+1} P_{i+1,n+1}(x)$$

And

$$\left(1 - \frac{x}{L}\right) P_{i,n}(x) = \frac{L-x}{L} P_{i,n}(x)$$

$$= \frac{L-x}{L} . \frac{1}{(L)^n} \binom{n}{i} (x)^i (L-x)^{n-i}$$

$$= \frac{\binom{n}{i}/\binom{n+1}{i}}{(L)^{n+1}} \binom{n+1}{i} (x)^i (L-x)^{n+1-i}$$

$$= \frac{\binom{n}{i}}{\binom{n+1}{i}} P_{i,n+1}(x) = \frac{n-i+1}{n+1} P_{i,n+1}(x)$$

By summing both sides of these expression, we have desired result.

Theorm (2 − 2)

The derivative of the degree generalized Bernstein basis polynomials are given by:

$$\frac{d}{dx} P_{i,n}(x) = \frac{n}{L}\left[P_{i-1,n-1}(x) - P_{i,n-1}(x)\right]$$

Proof

By using Definition $(1-4)$, this expression can be obtained as:

$$\frac{d}{dx} P_{i,n}(x) = \frac{d}{dx}\left(\frac{1}{L^n}\binom{n}{i} (x)^i (L-x)^{n-i}\right)$$

$$= \frac{1}{L^n}\binom{n}{i} \left[i(x)^{i-1}(L-x)^{n-i} - (n-i)(x)(L-x)^{n-i-1}\right]$$

$$= \frac{n}{L^n}\left[\binom{n-1}{i-1} (x)^{i-1}(L-x)^{n-i} - \binom{n-1}{i} (x)^i (L-x)^{n-i-1}\right]$$

$$= \frac{n}{L}\left[\binom{n-1}{i-1} \frac{1}{L^{n-1}} (x)^{i-1}(L-x)^{n-1-(i-1)} \binom{n-1}{i} \frac{1}{(L)^{n-1}} x^i (L-x)^{n-1-i}\right]$$

$$= \frac{n}{L}\left[P_{i-1,n-1}(x) - P_{i,n-1}(x)\right]$$

Theorm (2 − 3)

The first derivatives of the nth –degree generalized Bernstein basis polynomials of degree n:

$$P_{i,n}(x) = \frac{1}{L}\left[(n-i+1)P_{i-1,n}(x) + (2i-n)P_{i,n}(x) - (i+1)P_{i+1,n}(x)\right]$$

Proof

By utilizing theorem$(2-1)$, the following equation can be written as

$$P_{i,n-1}(x) = \frac{n-1}{n} P_{i,n}(x) + \frac{i+1}{n} P_{i+1,n}(x$$

$$P_{i-1,n-1}(x) = \frac{n-1+1}{n} P_{i-1,n}(x) + \frac{i}{n} P_{i,n}(x)$$

Substituting these relation into the right hand side of the expression theorem $(2-2)$, the desired relation is obtained.

Theorm $(2-3)$

There is relation between generalized Bernstein basis polynomials matrix and their derivatives of the form

$$P^{(k)}(x) = P(x)N^k, \qquad k = 1, 2, \dots, m$$

Here the elements of $(n+1) \times (n+1)$ matrix $N = (m_{ij}), i, j = 0, 1, \dots, n$ are defined by

$$m_{ij} = \frac{1}{L} \begin{cases} n-i & if \quad j = i+1 \\ 2i-n & if \quad j = 1 \\ -i & if \quad j = i-1 \\ 0 & 0.w. \quad . \end{cases}$$

Proof

From theorem $(2-3)$ and condition $P_{i,n}(x) = 0 \; if \; i < 0 \; or \; i > 0$, we have

$$\acute{p}_{0,n}(x) = \frac{1}{L} [-nP_{0,n}(x) - P_{1,n}(x)]$$

$$\acute{p}_{1,n}(x) = \frac{1}{L} [nP_{0,n}(x) + (2-n)P_{1,n}(x) - 2P_{2,n}(x)]$$

$$\acute{p}_{2,n}(x) = \frac{1}{L} [(n-1)P_{1,n}(x) + (4-n)P_{2,n}(x) - 3P_{3,n}(x)]$$

$$\vdots$$

$$\acute{p}_{n-1,n}(x) = \frac{1}{L} [2P_{n-2,n}(x) + (n-2)P_{n-1,n}(x) - nP_{n,n}(x)]$$

$$\acute{p}_{n,n}(x) = \frac{1}{L} [P_{n-1,n}(x) + nP_{n,n}(x)]$$

We have the matrix relation ṕ(x)= p(x)N

Such that

$$P(x) = [P_{0,n}(x), P_{1,n}(x), \dots, P_{n,n}(x)]$$

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$$\acute{p}(x) = \left[\acute{p}_{0,n}(x), \acute{p}_{1,n}(x), \dots, \acute{p}_{n,n}(x)\right]$$

$$N=\frac{1}{L}\begin{bmatrix} -n & n \dots & 0 & 0 & 0 \\ -1 & 2-n \dots & 0 & 0 & 0 \\ 0 & -2 \dots & 0 & 0 & 0 \\ 0 & 0 \dots & n-4 & 2 & 0 \\ 0 & 0 \dots & 1-n & n-2 & 1 \\ 0 & 0 \dots & 0 & -n & n \end{bmatrix}$$

In a similar way, the second derivative becomes

$$P^{n}(x) = \acute{p}(x)N = P(x)N^{2}$$

Thus we get derivatives of the generalized Bernstein basis polynomials in the form

$$P^{(k)}(x) = P^{(k-1)}(x)N = P(x)N^{k}$$

This completes the proof

$3 - Method\ of\ solution$

The main idea of the Bernstein collocation method is to seek a solution of the problem in the form of the Bernstein polynomials.

$Theorm\ (3-1)$

Let $x_i \epsilon [0, L]; i = 0, 1, \dots, n$ be collocation points. General mth –order linear non- homogeny differential equation of fractional order $(1-1)$ can be written as the matrix form:

$$P(x). N^{m}U - aP(x)N^{\propto}U - b\,U = f$$

Here the matrices are U=$\left[u\left(\frac{Li}{n}\right)\right]$; $P = \left[P_{0,n}(xi)\right]$ and $F = [f(xi)]$; $i, j = 0, 1, \dots, n$

$Lemma\ (3-2)$

Let the generalized Bernstein basis polynomials be defined on the interval [0, L]. then we have

$$\sum_{i=0}^{n} iP_{i,n}(x) = \frac{nx}{L}$$

$$\sum_{i=0}^{n} i^{2}P_{i,n}(x) = \frac{n(n-1)x^{2}}{L^{2}} + \frac{nx}{L}$$

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Proof

From property (b) and definition $(1-4)$ desired expression can be written as respectively

$$\sum_{i=0}^{n} i\, P_{i,n}(x) = \frac{1}{L^n}\sum_{i=0}^{n} i\binom{n}{i}(x)^i(L-x)^{n-i}$$

$$= \frac{1}{L^n}\sum_{i=0}^{n}\frac{n!}{(i-1)!(n-i)}(x)^i(L-x)^{n-i}$$

$$= \frac{nx}{L}\sum_{i=0}^{n}\frac{1}{(L)^{n-1}}\binom{n-1}{i}(x)^i(L-x)^{n-1-i}$$

$$= \frac{nx}{L}\sum_{i=0}^{n} P_{i,n-1}(x)$$

$$= \frac{nx}{L},$$

$$\sum_{i=0}^{n} i^2 P_{i,n}(x) = \frac{1}{L^n}\sum_{i=0}^{n} i^2\binom{n}{i}(x)^i\,(L-x)^{n-i}$$

$$\frac{n}{L^n}\left[\sum_{i=0}^{n}\frac{(i-1)(n-1)!}{(i-1)!(n-1)!}(x)^i(L-x)^{n+i} + \sum_{i=0}^{n}\frac{(n-1)!}{(i-1)!(n-i)!}(x)^i(L-x)^{n-i}\right]$$

$$= \frac{n(n-1)x^2}{L^2}\sum_{i=0}^{n} P_{i,n-2}(x) + \frac{nx}{L} - \sum_{i=0}^{n} P_{i,n-1}(x) = \frac{n(n-1)x^2}{L^2} + \frac{nx}{L}$$

4 – Numerical Result

The numerical example is considered by the using the presented method on collocation points $X_i = \frac{L}{n}, i = 1, 2, \dots, n$ and solved using other numerical schemes. This allows one to compare the results obtained using this scheme with the analytical solution or the solution obtained using other schemes.

Numerical result is written by using the symbolic calculus software mathematical.

Example (4 – 1)

Consider the following composite fractional relaxation

$$\frac{du}{dt} - a\frac{d^{\propto}u}{dt^{\propto}} - b = 0, t > 0, 0 < \propto \leq 1$$

Subject to initial condition u(0)=1

The numerical values when x=0.25 0.5 , 0.75, and a=0, b=L for equation above

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t	α=0.25		α=0.5		α=0.75	
	UFDM	Upresented method	UFDM	Upresented method	UFDM	Upresented method
0.0	1.0	1.0	1.0	1.0	1.0	1.0
0.1	0.7522	0.7515	0.6686	0.6621	0.6236	0.5669
0.2	0.6063	0.6057	0.5491	0.5436	0.5433	0.4939
0.3	0.4998	0.4993	0.4638	0.4638	0.4878	0.4434
0.4	0.4182	0.4178	0.4081	0.4040	0.4442	0.4038
0.5	0.3540	0.3537	0.3604	0.3569	0.4079	0.3708
0.6	0.3026	0.3023	0.3185	0.3185	0.0376	0.3426
0.7	0.2609	0.2607	0.2894	0.2865	0.3497	0.3179
0.8	0.2267	0.2265	0.2621	0.2594	0.3256	0.2960
0.9	0.1984	0.1981	0.2386	0.2363	0.3046	0.2764
1.0	0.1747	0.1795	0.2184	0.2162	0.2846	0.2587

This table shows the approximate solutions for eq. obtained for different values of X using the presented method compaction with fractional difference method.

We show that the presented method converges rapidly to exact solution of the linear differential equations for increasing x and t.

5 – *conclusions*

in this study, by using shifted Bernstein pseudo –spectral collocation method, linear differential equations under the initial conditions are expressed as a sequence of the linear equation of fraction order iteratively. Then, a collocation method based on generalized shifted Bernstein pseudo –spectral collocation method is developed for solving these equations.

The numerical result have been presented for showing applicability.

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